AN INFORMAL INTRODUCTION TO FORMAL GROUPS

DORON GROSSMAN-NAPLES

ABSTRACT. Infinitesimal objects in algebraic geometry have a rich structure, but they are difficult to study due to the failure of classical Lie theory in the algebraic context, especially in characteristic p. This failure can be attributed to the fact that Lie algebras only capture first-order infinitesimal behavior, a limitation which vanishes when we shift our focus to a new kind of infinitesimal object: formal groups. In this talk, I will describe the basic theory and examples of formal groups, as well as how they give rise to a surprising and deep connection between algebraic geometry and algebraic topology: chromatic homotopy theory.

CONTENTS

1. FROM DERIVATIVES TO POWER SERIES

In differential geometry, we have an adjunction $\mathrm{LieGp} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{LieAlg}$ which interpolates between local and global symmetries. Lie's theorems tell us that local smooth representation theory can be completely described by Lie algebras. This is no longer true for algebraic groups. Some of it can be recovered in characteristic 0, but it fails drastically in characteristic p. Why? Simple: the Lie algebra doesn't contain enough information, because it only has first-order infinitesimals.

It is a theorem following directly from the Leibniz rule that in characteristic 0, the associative algebra of differential operators at a point (in the sense of Grothendieck) is generated by the regular germs and first-order differential operators $\frac{\partial}{\partial x_i}$. This *fails* in positive characteristic, however. For example, over any field k, there is an n^{th}-order differential operator D^n at 0 on $\mathbb{A}^1(k)$ given by $D^n(x^{n+1}) = 1$. In characteristic 0, we can just write $D^n = \frac{1}{n!}(\frac{d}{dx})^n$. If $n = p$ and $\operatorname{char} k \leq p$, however, this fraction isn't defined! The operator still exists, though. Operators like this can be constructed as "divided power differential operators", but they're tricky to work with; and, in any case, their behavior isn't captured by the Lie algebra of a group. Solution: include higher-order infinitesimals intrinsically! This is the origin of the theory of formal groups.

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A Lie algebra describes the algebraic properties of first-order differential operators, which are dual to linear functions, so the natural way to extend to higher-order infinitesimals is to take formal groups to be dual to formal power series (hence the name). The bracket of vector fields is the universal operation on objects which preserve the local algebraic structure of linear functions; and power series have their own local algebraic structure to preserve. Note that for power series of the form $f(t) = t + a_1t^2 + a_2t^3 + \dots$, we have a formal composition operation. This looks something like $(t + \alpha_1 t^2 + \alpha_2 t^3 + \dots) \circ (t + \beta_1 t^2 + \beta_2 t^3 + \dots) = t + (\alpha_1 + \beta_1)t^2 + (\alpha_2 + 2\alpha_1\beta_1 + \beta_2)t^3 + \dots$. (The coefficients are not binomial coefficients, by the way; be careful.) The fact that the constant term is zero guarantees that the composition converges in the Krull topology, so this is a well-defined operation. It is "obviously" associative, it has identity t, and the existence of inverses is guaranteed by the Weierstrass preparation theorem. Therefore, they form a group, and this is the structure we want to dualize.

Definition 1.1. Let R be a ring. A *one-dimensional formal group law* on R is a formal power series $F(x, y) \in R[[x, y]]$ such that

i) $F(x, y) = x + y \pmod{(x, y)^2}$ and
ii) $F(x, F(y, z)) = F(F(x, y), z)$.

If $F(x, y) = F(y, x)$, we say that the formal group law is *commutative*.

An *n-dimensional formal group law* is defined similarly as a formal power series $F(x_1, \dots, x_n, y_1, \dots, y_n)$ over R satisfying analogous axioms.

Remark 1.2. The existence of inverses follows automatically from these axioms. To be precise, there exists G such that $F(x, G(x)) = F(G(x), x) = 0$.

The idea of this definition is that F is supposed to be the power series representing the multiplication operation in some geometric group. The power series composition group defined above describes the multiplication in the infinite-dimensional group of "formal automorphisms at 0" of whatever scheme we're working on. This is essentially the universal "formal change of coordinates group", as we will see later.

Some examples: the simplest possible example is the additive formal group law, $F(x, y) = x + y$. This should be thought of as the infinitesimalization of the additive group $\mathbb{G}_a(R) = (\mathbb{A}^1(R), +)$ at 0. Similarly, the multiplicative formal group law, $F(x, y) = x + y + xy$, is the infinitesimalization at 1 of the multiplicative group $\mathbb{G}_m(R) = (R^*, \cdot)$. The relationship between these two laws is essentially a logarithm; this is literally true in the case of Lie groups, where there is an isomorphism between them induced by the exponential/log isomorphism. We will see that this is in fact true more generally when interpreted appropriately: the universal one-dimensional commutative formal group law is obtained from the additive formal group law by changing coordinates using the exponential. More on that later. One last example (from number theory) is the Lubin-Tate formal group law on the p-adic integers $\hat{\mathbb{Z}}_p$, which is the unique (up to "strict isomorphism") formal group law such that the map $x \mapsto px + x^p$ is an endomorphism. This formal group law describes the deformation theory of ramified extensions, facilitating the construction of abelian extensions of number fields in local class field theory. (This is also connected to Morava E-theory in stable homotopy—stay tuned.)

2. Formal Geometry

So far, I've given you a description of formal groups in coordinates. Much like algebraic groups, however, they also have an intrinsic geometric description, and this description is crucial to describing them in full generality. Therefore, to understand formal groups in the language of schemes, I'll now betray the word "informal" in the title of my talk and categorify. For the rest of the talk, I will follow standard practice in formal geometry and assume all rings to be Noetherian.

Definition 2.1. An *adic ring* is a topological ring R which carries some Krull topology (the topology generated by translations of powers of some ideal), and which is complete and Hausdorff.

Remark 2.2. Note that the ideal defining the topology is not part of the structure. In general, multiple ideals of a complete ring can give rise to the same Krull topology—take I and I^2, for instance.

Adic rings are a generalization of power series rings with their usual Krull topology. In fact, by the Cohen structure theorem, any *regular local* adic ring is guaranteed to be a power series ring provided it is equicharacteristic (i.e. contains a field). If it is of mixed characteristic, it is still a power series ring (though over a DVR rather than a field) as long as it is unramified.

For any adic ring R, we can define an associated locally ringed space $\operatorname{Spf} R$, the *formal spectrum* of R, which is $\varinjlim \operatorname{Spec}(R/I^n)$. This colimit is taken in locally ringed spaces, and while this particular construction involves a choice of generating ideal I, the formal spectrum is independent of this choice; it's just a convenient neighborhood system. This construction, which is the geometric analogue of the completion of a ring, can in fact be applied this to any sheaf of ideals on a scheme to get a "formal scheme" ([3], II.9). Such objects can be described as ind-schemes (formal filtered colimits of schemes) and realized as presheaves in a generalized functor-of-points approach. Today, however, we'll view them as locally ringed spaces.

Definition 2.3. A *formal scheme* is a locally ringed space which is locally isomorphic to the formal spectrum of an adic ring. If it is locally isomorphic to the formal spectrum of a formal power series ring, we call it a *formal Lie variety.*

Warning 2.4. It's important to note that not every formal scheme is a formal completion of an ordinary scheme. Formal schemes which can be obtained in this way are called *algebraizable.*

This definition actually includes all ordinary schemes as well by completing at the trivial sheaf (as we would expect, since these are supposed to be pro-schemes). As in the classical case, we can talk about relative formal schemes, and in particular the category of formal schemes over a ring R. We'll be particularly interested in formal Lie varieties over affine schemes, since they give us a geometric description of formal groups[1].

Definition 2.5. A *formal group* over R is a group object in the category of formal Lie varieties over R.

[1]Note that formal Lie varieties are actually relatively sparse among formal schemes. For instance, the completion at a closed subscheme Y is a formal Lie variety if and only if Y is a 0-dimensional regular scheme which is equicharacteristic or unramified of mixed characteristic.

Given this definition, it's not hard to verify that an *affine* formal group is the same as a formal group law. More precisely, we have the following result.

Theorem 2.6. *Define an n-dimensional formal group law on a ring R to be an R-affine formal group of relative dimension n, i.e. a formal group over R whose underlying relative formal scheme is* $\mathrm{Spf}(R[[x_1, \ldots, x_n]])$*. Then this definition coincides with the definition in coordinates given above.*

With this generalization, we can finally give a rigorous description of how formal groups interpolate between geometric groups and their Lie algebras. The Lie algebra of an n-dimensional formal group law F is defined to be R^n with bracket $[\mathbf{x}, \mathbf{y}] = F_2(\mathbf{x}, \mathbf{y}) - F_2(\mathbf{y}, \mathbf{x})$. This definition extends to all formal groups by taking local coordinates. On the other hand, the formal group of a smooth and unramified[2] algebraic group G is defined simply to be the formal completion of G at the identity. This is a rich source of formal groups, and gives rise to things like elliptic cohomology.

Theorem 2.7. *The composite functor* $\mathrm{GpSch}_{/R} \to \mathrm{FG}_R \to \mathrm{LieAlg}_R$ *is the ordinary Lie algebra functor.*

3. Classification of Formal Groups

Many of the results of this and the next section can be found in [1].

Now we have an appropriate notion of formal group, we'd like to know about their structure. As promised, this structure coincides with that of Lie algebras in characteristic 0.

Theorem 3.1. *Let R be a ring of characteristic* 0*. Then the Lie algebra functor* $\mathrm{FG}_R \to \mathrm{LieAlg}_R$ *is an equivalence of categories.*

Proof. The idea of the proof is to give an inverse functor, and this functor is given by the Baker-Campbell-Hausdorff formula from Lie theory: $e^X e^Y = e^Z$, where Z is given by a power series $Z = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] + \frac{1}{12}[[X, Y], Y] + \ldots$. This describes the group multiplication, and points of the formal group are just formal exponentials of Lie algebra elements. □

Although this result only applies in characteristic 0 due to the division required, the notion of exponential does generalize in a useful way.

Theorem 3.2. *For each $n \in \mathbb{N}$, there is a universal commutative n-dimensional formal group (R, F). That is, R represents the cocartesian fibration* $\mathrm{CFGL}_n \to \mathrm{CRing}$*, with a map $f : R \to S$ inducing the formal group law $f_*(F)(\boldsymbol{x}, \boldsymbol{y}) \in S[[\boldsymbol{x}, \boldsymbol{y}]]$.*

I won't describe the proof in detail (though see below for the case $n = 1$), but suffice to say that one can take a free algebra with relations describing the axioms of a formal group law, which is equivalent to an explicit construction in terms of Witt vectors; see ([4], section 11) for details.

Of particular interest is the one-dimensional case.

[2]By "unramified", I mean that the map $\mathrm{Spec}(\kappa(e)) \to G$ classifying the identity is an unramified map of schemes. This is a geometrization of the ramification condition of the Cohen structure theorem.

Definition 3.3. Write $P = \mathbb{Z}[a_{ij}]_{i,j\in\mathbb{Z}^+}$, and define a formal power series $\mu(x,y) = x + y + \sum a_{ij}x^iy^j \in P[[x,y]]$. Then we can write $\mu(x,\mu(y,z)) - \mu(\mu(x,y),z) = \sum b_{ijk}x^iy^jz^k$ for some coefficients $b_{ijk} \in P$. Let I be the ideal generated by all b_{ijk} and $a_{ij} - aji$. Then the *Lazard ring* is defined as $L = P/I$.

The Lazard ring with $\bar{\mu}$ is the universal one-dimensional commutative FGL, a fact which follows directly from the definition. However, as given, the actual structure of L is complicated to understand. The following result, therefore, is quite miraculous.

Theorem 3.4 (Lazard)**.** *The Lazard ring L is isomorphic to a polynomial algebra on infinitely many generators: $L \cong \mathbb{Z}[b_1, b_2, \dots]$.*

This isomorphism is given as follows, where by convention $b_0 = 1$. Writing R for the polynomial ring, define a power series $\exp \in R[[x]]$ by $\exp(y) = \sum_{i\in\mathbb{N}} b_iy^{i+1}$. Then exp is invertible with respect to composition, so write log for its inverse. Define a formal group law on R by $\mu^R(x,y) = \exp(\log(x)+\log(y))$. This is obviously one-dimensional and commutative, and the classifying map $L \to R$ for this FGL turns out to be an isomorphism. The proof, as well as explicit formulae for the coefficients of log, can be found in ([1], section 7). This result can be interpreted as saying that any one-dimensional commutative formal group law is built from an additive formal group law. In fact, in the characteristic 0 case, we can actually transfer this description from the universal FGL to any particular FGL, which recovers the BCH trivialization from above.

This gives us a classification of (1d commutative) formal group *laws*, but what about formal *groups*? For that, we have to descend[3] to stacks. Recall from earlier that we have an action of the composition group scheme of formal power series $G = \mathrm{Spec}(\mathbb{Z}[b_1, b_2, \dots])$ on formal group laws; for a point $f = t + b_1t^2 + \dots$, the action is given by $f \cdot F(x,y) = f(F(f^{-1}(x), f^{-1}(y)))$. I called this the "universal formal change of coordinates group", which is true for formal group laws, but it needs to be modified for the non-affine case. Accordingly, we enhance this slightly to the group scheme $G^+ = \mathbb{G}_m \rtimes G$ of formal power series $b_0t + b_1t^2 + \dots$ with the obvious action. Then we have the following (cf [5], lecture 11):

Definition 3.5. The *moduli stack of formal groups* is the quotient stack $\mathcal{M}_{FG} = \mathrm{Spec}(L)/G^+$.

Theorem 3.6. *The moduli stack of formal groups with $\bar{\mu}$ is the universal formal group. That is, the cartesian fibration* $\mathrm{CFG}_1 \to \mathrm{CRing}$ *is represented by $\mathcal{M}_{FG}$.*

It is not known whether $\mathcal{M}_{FG}$ is a Deligne-Mumford stack. It is, however, stratified by Deligne-Mumford substacks $\mathcal{M}_{FG}^n$. A formal group in the n^{th} stratum is said to have *height n*. Writing $\bar{+}$ for the formal group law, a formal group over a ring of characteristic p has height at least n iff the coefficient v_i of x^{p^i} in the p-fold formal sum $x\bar{+}\dots\bar{+}x$ is 0 for $i < n$, and height exactly n if it has height at least n and v_n is invertible.

[3]Pun intended.

4. The Music of the Spheres

> My initial inclination was to call this book The Music of the Spheres, but I was dissuaded from doing so by my diligent publisher, who is ever mindful of the sensibilities of librarians.
>
> Doug Ravenel, [6]

The biggest reason for people's interest in formal groups today is undoubtedly the peculiar connection between algebraic topology and algebraic geometry known as chromatic homotopy theory. An introduction to the chromatic point of view would be a talk in itself, but I'll briefly summarize what's going on here.

The main goal of stable homotopy theory is to compute the stable homotopy groups of spheres $\pi_*(\mathbb{S})$. A typical way to do this is using an Adams spectral sequence. This is a spectral sequence abutting to the (p-completed) stable homotopy groups of spheres, or more generally to the graded hom group between any two spectra. This spectral sequence arises by a filtration coming from "killing off cohomology", one dimension at a time (the Adams resolution); this can be thought of as killing off the cells of a CW complex one dimension at a time. Because this is described cohomologically, one can generalize and replace ordinary $\mathbb{F}_p$ cohomology with other multiplicative cohomology theories, and in particular with complex cobordism, yielding the Adams-Novikov spectral sequence. This spectral sequence computes the stable homotopy groups in their entirety, but is correspondingly more complicated. This is where the connection to formal groups enters in. (All formal groups are henceforth assumed one-dimensional and commutative.)

Theorem 4.1. *The homotopy ring of complex cobordism is the Lazard ring:* $\pi_*(MU) \cong L$ *with grading given by* $\deg(b_i) = 2i$.

We find that any complex-oriented cohomology theory (E^∞ ring spectrum with a ring map from MU[4]) therefore admits a formal group law on its homotopy ring. This can also be described explicitly: $\mathbb{CP}^\infty$ has a group structure in Ho(Top) arising from the tensor product of complex line bundles. Applying a complex-oriented cohomology theory E to $\mathbb{CP}^\infty$ yields a power series ring over $\pi_*(E)$ in one variable (the generalized Chern class), and the induced product on this ring defines a formal group law on $\pi_*(E)$.

Conversely, given a formal group law (R, F), one can define a functor $MU \otimes_L R$; but, since R need not be flat, this may fail to be an actual cohomology theory. The Landweber Exact Functor Theorem gives a criterion for this functor to in fact be a cohomology theory.

Theorem 4.2 (Landweber)**.** *Let* (R, F) *be a formal group law, and for each prime number* p *let* v_i *be as defined above. If* $v_0, v_1, v_2, \ldots, v_n$ *is a regular sequence in* R *for each* p *and each* n, *then* $MU \otimes_L R$ *is a cohomology theory.*

Using this theorem, we can lift the stratification of $\mathcal{M}_{FG}$ to a filtration in the world of topology. The deformation theory of the stratification is described by a generalized form of the Lubin-Tate formal group law, and these formal group laws classify (via LEFT) a height-ordered collection of complex-oriented cohomology theories called Morava E-theories. This is where the term "chromatic" comes from: much how like light can be understood by splitting it into different colors in the

[4] This can be thought of as a generalized Chern class.

chromatic spectrum, or how musical notes can be classified by dividing them into a discrete set of notes in the chromatic scale, splitting complex-ordered cohomology theories by height allows us to separate the different patterns that show up in the stable homotopy groups of spheres. Localizing sequentially at Morava E-theories gives a height-based filtration called the "chromatic tower", and the associated "chromatic" spectral sequence converges to the E_2 page of the Adams-Novikov spectral sequence. Replacing $\mathcal{M}_{FG}$ with its analogue in spectral algebraic geometry, the nonconnective spectral moduli stack of oriented formal groups, actually yields the *entire* Adams-Novikov spectral sequence ([2])—but that's another story.

References

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